Nonhomogeneous Rope Modes and Infinite Time Domain Approximations

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Abstract

This paper has two significant results. First, it details how modes of the simple wave equation (the equation that dictates rope movement under high tension and small displacement) on a finite spatial domain can be solved for, when the density function of the rope is arbitrary. Secondly, it details how one can use this sequence of modes to approximate an arbitrary wave pulse on the rope. Once the wave pulse is approximated as a sum of modes, one can plug in any time value and get a result without having to simulate every second beforehand.

1 Foreword

This article is based off of part of chapter 5 of DeVries' A First Course in Computational Physics (2ed). While it's meant to be self-contained it forms more of a documentation of work than a readable/nice paper. See the book by DeVries or the visualizations on my website to get a more complete idea of this project. My website also contains a .pdf and a Mathematica .nb file giving all Mathematica code used to solve this problem.

Explicitly: Code, visualizations, and larger motivation/descriptions are omitted in this paper.

2 Introduction

We solve the 1-dimensional wave equation with fixed boundaries by applying the finite element method as described in DeVries 5.26. The full time dependent one dimensional partial differential equation can be described as follows, where $\phi(x,t)$ is the height of the wave and is a function of position and time, $\mu(x)$ is the density of the wave medium at position x, and T is the tension on the wave medium:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2\psi}{\partial t^2} \tag{1}$$

We can solve for the normal modes of this system by assuming the height function ψ is of the form $\psi = ay(x)\cos(\omega t + b)$ for nonunique constants a, b, and ω. Inserting this into equation 1:

$$\begin{aligned}0 &= \frac{\partial^2\psi}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2\psi}{\partial t^2} \\ &= ay''(x)\cos(\omega t + b) - \frac{\mu}{T}(-ay(x)\omega^2\cos(\omega t + b))\end{aligned}$$

The final form reached is a time independent differential equation for $y(x)$:

$$0 = y'' + \frac{\mu}{T}y\omega^2 \tag{2}$$

Depending on the function $\mu(x)$, this differential equation may not have a simple solution.

3 Finite Elements Equations

We will assume nothing about $\mu(x)$ (except that it is reasonably well behaved and non-pathological). The solution to 2 can then only be calculated numerically.

To approximate y, assume that the domain in question ranges from $x = 0$ to $x = L$. To implement fixed boundary conditions, it will be assumed that $y(0) = y(L) = 0$. This domain is divided into $N + 1$ evenly spaced points indexed from 0 to N, with $x_0 = 0$ and $x_N = L$. Then the separation between consecutive x_i is the step size and will be denoted:

$$h = x_{i+1} - x_i = \frac{L}{N} \tag{3}$$

Note that this definition sets $x_i = hi$.

The function y will be linearly approximated by $N + 1$ constants a_i, with the property that $y(x_i) = a_i$, and linear interpolation is used between points. This can be described in a convenient manner by the use of basis functions. If one writes:

$$\phi_i(x) = \begin{cases} 0, & \text{if } x \le x_{i-1} \\ \frac{x - x_{i-1}}{x_i - x_{i-1}}, & \text{if } x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x_{i-1}}{x_i - x_{i-1}}, & \text{if } x_i \le x \le x_{i+1} \\ 0, & \text{if } x_{i+1} \le x \end{cases} \tag{4}$$

then p can be written as

$$p(x) = \sum_{i=0}^{N} a_i\phi_i(x) \tag{5}$$

Equation 2 contains a second derivative, so it is immediately applicable to consider ϕ_i' and ϕ_i''. We will address the discontinuities by applying the standard methods of the Heaviside step function and the Dirac delta function. We find that ϕ_i can be written in the cumbersome form, which is equivalent to 4:

$$\phi_i(x) = \frac{1}{h}((x - x_{i-1})H(x - x_{i-1}) - 2(x - x_i)H(x - x_i) + (x - x_{i+1})H(x - x_{i+1})) \tag{6}$$

In this equation, $H(a)$ is the Heaviside step function and is 0 when $a < 0$ and 1 when $a > 1$. Using standard methods, the derivative of this function is defined in terms of distributions, and after simplifications it comes out to be

$$\phi_i'(x) = \frac{1}{h}(H(x - x_{i-1}) - 2H(x - x_i) + 1H(x - x_{i+1})) \tag{7}$$

Applying the same process a second time:

$$\phi_i''(X) = \frac{1}{h}(\delta(x - x_{i-1}) - 2\delta(x - x_i) + \delta(x - x_{i+1})) \tag{8}$$

where δ is the Dirac delta distribution.

The residual error, the amount by which p does not satisfy the differential equation, is given by $R(x) = \frac{d^2p}{dx^2} + \mu(x)\frac{\omega^2}{T}p$. Applying the Galerkin method yields the condition that $\int_0^L R(x)\phi_i(x)dx = 0$, for all i.

$$\begin{aligned}
0 &= \int_0^L (\frac{d^2p}{dx^2}\phi_i(x) + \mu(x)\frac{\omega^2}{T}p\phi_i(x))dx \\
&= \int_0^L \left(\sum_{n=0}^{N} \frac{a_n}{h}\left(\delta(x - x_{n-1}) - 2\delta(x - x_n) + \delta(x - x_{n+1})\right)\phi_i(x)\right) dx \\
&+ \int_0^L (\sum_{n=0}^{N} \mu(x)\frac{\omega^2}{T}a_n\phi_i(x)\phi_n(x))dx
\end{aligned} \tag{9}$$

This integral is explicitly integrable if we approximate $\mu(x)$ as constant through the interval (x_i, x_{i+1}), and denote it μ_i. Doing this reveals that equation 9 is equivalent to:

$$\begin{aligned}
0 = \sum_{n=0}^{N} \Bigg(&\frac{a_n}{h}\int_0^L \left(\delta(x - x_{n-1}) - 2\delta(x - x_n) + \delta(x - x_{n+1})\right)\phi_i(x)dx \\
&+ \mu_i\frac{\omega^2}{T}a_n\int_0^L \phi_i(x)\phi_n(x)dx\Bigg)
\end{aligned} \tag{10}$$

Due to the localized nature of the basis functions, both terms in the sum only become nonzero when $n = i - 1$, $n = i$ or $n = i + 1$. So, the integrals can be evaluated explicitly, and

$$\omega^2\left(a_{i-1} + 4a_i + a_{i+1}\right) = \frac{6T}{\mu_i h^2}\left(a_{i-1} - 2a_i + a_{i+1}\right) \tag{11}$$

4 Approximating a function as a sum of modes

If we are given the initial conditions of the rope at time 0 as $\psi(x, 0) = f(x)$ and $\frac{\partial}{\partial t}\psi(x, 0) = \dot{f}(x)$, we can approximate the rope's evolution over time by representing the equation as a finite sum of M different normal modes. We will denote the ith normal mode with frequency ω_i as $g_i(x)$, where g_i is a solution to equation 2.

Suppose that $\Gamma(x,t)$ is a function which is the sum of a finite number of normal modes at time t. To best approximate ψ we will need an appropriate number of degrees of freedom. One selection of degrees of freedom is to have $2M$ constants, denoted a_i and b_i, where a_i is a magnitude and b_i is a phase. Writing out our equation for Γ and its time derivative and evaluating at time $t = 0$:

$$\Gamma(x,t) = \sum_{i=1}^{M} a_i \cos(\omega_i t - b_i) g_i(x) \tag{12}$$

$$\frac{\partial}{\partial t}\Gamma(x,t) = \sum_{i=1}^{M} g_i(x) a_i \omega_i (-\sin(\omega_i t - b_i)) \tag{13}$$

$$\Gamma(x,0) = \sum_{i=1}^{M} g_i(x) a_i \cos(b_i) \tag{14}$$

$$\frac{\partial}{\partial t}\Gamma(x,0) = \sum_{i=1}^{M} g_i(x) \omega_i a_i \sin(b_i) \tag{15}$$

Now we suppose that a_i are chosen such that Γ is the best possible least-squares approximation to $f(x)$ (it will be found that this condition imposes no restrictions on b_i). We then assume that b_i are chosen such that $\frac{\partial}{\partial t}\Gamma$ is the best possible least-squares approximation to $\dot{f}(x)$ (likewise this condition imposes no restrictions on a_i).

Minimization of the least-squares error is carried out for each a_i. Since we care about the initial conditions, we assume $t = 0$ so that both Γ and f are dependent on x but not t.

$$\begin{aligned} 0 &= \frac{\partial}{\partial a_i} \int_0^L (\Gamma - f)^2 dx \\ &= \frac{\partial}{\partial a_i} \int_0^L (\Gamma^2 + f^2 - 2f\Gamma) dx \\ &= \int_0^L (2\Gamma \frac{\partial}{\partial a_i}\Gamma - 2f \frac{\partial}{\partial a_i}\Gamma) dx \\ &= 2 \int_0^L (\Gamma - f) \frac{\partial \Gamma}{\partial a_i} dx \end{aligned} \tag{16}$$

Γ is expressed in equation 14, and clearly $\frac{\partial \Gamma}{\partial a_i} = g_i(x)\cos(b_i)$. Since $\cos(b_i)$ is a constant term and not a function of x, we can throw it away (assuming the input function f is well behaved enough that $\cos(b_i)$ can be chosen not equal to zero) along with the factor of two. Doing that and moving the integral of f to the other side of the equation turns equation 16 into

$$\sum_{n=1}^{M} a_n \cos(b_n) \int_0^L g_n(x) g_i(x) dx = \int f(x) g_i(x) dx \tag{17}$$

Now we handle b_i similarly:

$$\begin{aligned} 0 &= \frac{\partial}{\partial b_i} \int_0^L (\dot{\Gamma} - \dot{f})^2 dx \\ &= \frac{\partial}{\partial b_i} \int_0^L (\dot{\Gamma}^2 + \dot{f}^2 - 2\dot{f}\dot{\Gamma}) dx \\ &= \int_0^L (2\dot{\Gamma} \frac{\partial}{\partial b_i} \dot{\Gamma} - 2\dot{f} \frac{\partial}{\partial b_i} \dot{\Gamma}) dx \\ &= 2 \int_0^L \left(\dot{\Gamma} - \dot{f} \right) \frac{\partial \dot{\Gamma}}{\partial b_i} dx \end{aligned} \tag{18}$$

$\dot{\Gamma}$ is expressed in equation 15, with $\frac{\partial \dot{\Gamma}}{\partial b_i} = g_i(x) a_i \omega_i \cos(b_i)$. Similar to last time, we assume $a_i \neq 0$ and $\omega_i \neq 0$ without justification. Reasonably but perhaps surprisingly, since $\cos(b_i)$ is a multiplying factor for the whole equation it is irrelevant to the answer (with the assumption $\cos(b_i) \neq 0$). So, both sides can be divided through by $2a_i\omega_i \cos(b_i)$. Finally, expanding $\dot{\Gamma}$ in terms of its sum and moving the $\dot{f}$ term to the other side turns equation 18 into:

$$\sum_{n=1}^{M} a_n \omega_n \sin(b_n) \int_0^L g_n(x) g_i(x) dx = \int \dot{f}(x) g_i(x) dx \tag{19}$$

In order to solve equations 17 and 19 we use linear algebra notation. It is not simplified to a sparse matrix and is left as a dense M by M linear equation. We define a matrix G_{ij}, and column vectors F_i, $\dot{F}_i$, C_i, and S_i.

$$G_{ij} = \int_0^L g_i(x) g_j(x) dx \tag{20}$$

$$F_i = \int_0^L f(x) g_i(x) dx \tag{21}$$

$$\dot{F}_i = \int_0^L \dot{f}(x) g_i(x) dx \tag{22}$$

$$C_i = a_i \cos(b_i) \tag{23}$$

$$S_i = \omega_i a_i \sin(b_i) \tag{24}$$

Then equations 17 and 19 - which both hold for all valid indices i - can be reexpressed in two matrix equations:

$$GC = F \tag{25}$$

$$GS = \dot{F} \tag{26}$$

Once C and S have been solved for, it is trivial to find a_i and b_i. Explicitly:

$$a_i = \sqrt{C_i^2 + (S_i \omega_i^{-1})^2} \tag{27}$$

$$b_i = \mathrm{atanfull}(S_i \omega_i^{-1}, C_i) \tag{28}$$